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QLS-Integrality of Complete r-Partite Graphs

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Abstract. A graph G is called A-integral (L-integral, Q-integral, S-integral) if the spectrum of its adjacency (Laplacian, signless Laplacian, Seidel) matrix consists entirely of integers. In this paper we study connections between the Q- (L,S,A) integral complete multipartite graphs. Moreover, new sufficient conditions for a construction of infinite families of QLS-integral complete r''-partite graphs $K_{p_1,p_2,\dots,p_{r''}} = K_{b_1\cdot p_1,b_2\cdot p_2,\dots,b_s\cdot p_s}$ from given QLS-integral r'-partite graphs $K_{p_1,p_2,\dots,p_{r'}} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ are given. Using these conditions new infinite classes of such graphs for $s = 4,5,6$ are constructed, which affirmatively answers to questions proposed by Wang, Zhao and Li in [10, 14]. Finally, we propose open problems for further study.

1. Introduction

We shall start with some definitions to a general M-theory.

Let G be a simple graph on n vertices, and let M be a real symmetric matrix associated to G. The characteristic polynomial $|xI - M|$ of M is called the M-characteristic polynomial (or M-polynomial) of G and is denoted by $M_G(x)$. The eigenvalues of M (i.e. the zeros of $M_G(x)$) are also called the M-eigenvalues of G (M-spectrum of G). The M-spectrum of G is real because M is symmetric.

In particular, if M is equal to one of the matrices A (adjacency matrix), $Q = D(G) + A$, $L = D(G) - A$, $S = J - I - 2A$, where $D(G)$ is the diagonal matrix of the vertex degrees in G and J is a square matrix with all elements equal to 1, then the corresponding spectrum is called A-spectrum, Q-spectrum, L-spectrum and S-spectrum, respectively. Throughout the paper the corresponding characteristic polynomials are denoted by $P_G(x) = |xI - A|$, $Q_G(x) = |xI - Q|$, $L_G(x) = |xI - L|$, $S_G(x) = |xI - S|$, respectively. The zeros of these polynomials are denoted by $\lambda_i; i = 1,2,\dots,n$, $\mu_i; i = 1,2,\dots,n$, $\kappa_i; i = 1,2,\dots,n$ and $\rho_i; i = 1,2,\dots,n$, respectively. A graph G is M-integral, $M \in \{A,Q,L,S\}$, if all the eigenvalues of its M-polynomial are integers. The study of integral graphs was initiated in [3]. A survey of integral graphs is given in [1]. For a connections between M theories see [2].

A complete r-partite graph $K_{p_1,p_2,\dots,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup \dots \cup V_r$ of $|V| = p_1 + p_2 + \dots + p_r$ vertices, where V_i's are nonempty disjoint sets, $|V_i| = p_i$ for $1 \le i \le r$, such that two vertices in V are adjacent if and only if they belong to different V_i's. Assume that the number of distinct integers of $p_1, p_2, \dots, p_r$ is s. Without loss of generality, assume that the first s ones are distinct integers such that $p_1 < p_2 < \dots < p_s$. The complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{p_1,\dots,p_1,\dots,p_s,\dots,p_s}$ is also denoted by $K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$

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and $|V| = \sum_{i=1}^{s} a_i p_i$. For results on A-integral complete r-partite graphs see for example [4–6, 9, 11–13]. For results on Q-integral complete r-partite graphs see for example [8, 14], for results on S-integral complete r-partite graphs see [7, 8, 10] and for results on L-integral complete r-partite graphs see [15].

In this paper we give a relationship for M-integrality of complete r-partite graphs where $M \in \{A, Q, L, S\}$. For example, it is easy to see that if $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ is Q-integral, then it is also S-integral and L-integral. We also give new sufficient conditions for a construction of infinite classes of Q-integral (S-integral) complete r''-partite graphs to a given Q-integral (S-integral) complete r'-partite graph. Using these conditions we construct infinite classes of QLS-integral complete multipartite graphs, which affirmatively answers to questions 4.1 and 4.2 of [10] and also questions 4.1 and 4.2 of [14]. Although concrete examples of QLS-integral complete multipartite graphs $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ with $s = 4, 5, 6$ are given in [8], in this paper we construct infinite classes of these graphs. Finally, we propose open problems for further study.

2. Preliminaries

In [14] Zhao et al. gave necessary and sufficient conditions for complete multipartite graphs $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ to be Q-integral, which are given in the following theorem.

Theorem 2.1. *[14] If the complete r-partite graph $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ on n vertices, $n = a_1 p_1 + a_2 p_2 + \cdots + a_s p_s$, is Q-integral, then there exists integers $\mu_i (i = 1, 2, ..., s)$ such that*

$$-\infty < n - 2p_s < \mu_s < n - 2p_{s-1} < \mu_{s-1} < \cdots < n - 2p_1 < \mu_1 < \infty \tag{1}$$

and the positive integers $a_i (i = 1, 2, ..., s)$ satisfying

$$a_k = \frac{\prod_{i=1}^{s} (\mu_i - n + 2p_k)}{p_k \prod_{i=1, i \neq k}^{s} 2(p_k - p_i)}; k = 1, 2, \ldots, s. \tag{2}$$

Conversely, suppose that there exist integers $\mu_i (i = 1, 2, ..., s)$ such that $-\infty < n - 2p_s < \mu_s < n - 2p_{s-1} < \mu_{s-1} < \cdots < n - 2p_1 < \mu_1 < \infty$ and the numbers

$$a_k = \frac{\prod_{i=1}^{s} (\mu_i - n + 2p_k)}{p_k \prod_{i=1, i \neq k}^{s} 2(p_k - p_i)}; k = 1, 2, \ldots, s$$

are positive integers. Then the complete r-partite graph $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ is Q-integral.

In [10] Wang et al. gave similar necessary and sufficient conditions for multipartite graphs $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ to be S-integral.

Theorem 2.2. *[10] If the complete r-partite graph $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ on n vertices is S-integral then there exists integers $\rho_i (i = 1, 2, ..., s)$ such that*

$$\rho_1 < 2p_1 - 1 < \rho_2 < 2p_2 - 1 < \cdots < 2p_{s-1} - 1 < \rho_s < 2p_s - 1 < \infty \tag{3}$$

and the numbers $a_1, a_2, \ldots, a_s$ satisfying

$$a_k = -\frac{\prod_{i=1}^{s} (\rho_i - 2p_k + 1)}{p_k \prod_{i=1, i \neq k}^{s} 2(p_i - p_k)}; k = 1, 2, \ldots, s \tag{4}$$

are positive integers.

Conversely, suppose that there exist integers $\rho_i (i = 1, 2, ..., s)$ such that $\rho_1 < 2p_1 - 1 < \rho_2 < 2p_2 - 1 < \cdots < 2p_{s-1} - 1 < \rho_s < 2p_s - 1 < \infty$ and the numbers

$$a_k = -\frac{\prod_{i=1}^{s}(\rho_i - 2p_k + 1)}{p_k \prod_{i=1, i\neq k}^{s} 2(p_i - p_k)}; k = 1, 2, \ldots, s$$

are positive integers. Then the complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1, a_2\cdot p_2,\ldots,a_s\cdot p_s}$ *is S-integral.*

The following theorem gives a relationship between Q-integrality and S-integrality of complete r-partite graphs.

Theorem 2.3. *[8] A complete multipartite graph is S-integral if and only if it is Q-integral.*

The theorem straightforward follows from the fact that for the nontrivial factors of the Q- and S-characteristic polynomials

$$Q^*(x) = \prod_{i=1}^{s}(x - n + 2p_i)\left(1 - \sum_{i=1}^{s}\frac{p_i}{x - n + 2p_i}\right)$$

and

$$S^*(x) = \prod_{i=1}^{s}(x - 2p_i + 1)\left(1 + \sum_{i=1}^{s}\frac{p_i}{x - 2p_i + 1}\right)$$

holds that

$$S^*(x) = (-1)^s Q^*(n - x - 1),$$

from which follows that Theorem 2.2 can be proved from Theorem 2.1 using substitution $\rho_i = n - \mu_i - 1$.

Theorem 2.4. *[15] The graph* $K_{p_1,p_2,\ldots,p_r}$ *is L-integral for every positive integers* p_i *and its L-spectrum is* $\{0, n^{r-1}, (n - p_i)^{p_i - 1}\}; i = 1, 2, \ldots, r$, *where n is the number of vertices of* $K_{p_1,p_2,\ldots,p_r}$.

Example 2.5. *We shall give here A-spectrum, Q-spectrum, L-spectrum and S-spectrum for two classes of graphs: complete graphs* K_{p_1} *and complete* a_1*-partite graphs* $K_{a_1\cdot p_1}$.

Complete graph $K_{p_1}, p_1 \geq 2$:

A: $\{p_1 - 1, (-1)^{p_1-1}\}$

Q: $\{2(p_1 - 1), (p_1 - 2)^{p_1-1}\}$

L: $\{0, p_1^{p_1-1}\}$

S: $\{p_1 - 1, (-1)^{p_1-1}\}$

Complete a_1*-partite graph* $K_{a_1\cdot p_1}$:

A: $\{p_1(a_1 - 1), (-1)^{a_1-1}, 0^{a_1p_1-a_1}\}$

Q: $\{2p_1(a_1 - 1), (a_1p_1 - 2p_1)^{a_1-1}, (a_1p_1 - p_1)^{a_1(p_1-1)}\}$

L: $\{0, a_1p_1^{a_1-1}, (a_1p_1 - p_1)^{a_1(p_1-1)}\}$

S: $\{p_1 - 1, (-1)^{a_1p_1-1}\}$

So K_{p_1} *and* $K_{a_1\cdot p_1}$ *are AQLS-integral for any* $a_1, p_1 \in N$.

3. Main Results

The following two theorems give constructions of the infinite class of Q-integral (S-integral) graphs $K_{b_1\cdot p_1, b_2\cdot p_2,\ldots,b_s\cdot p_s}$ from known Q-integral (S-integral) graph $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$. The similar theorem for A-integral graphs is given in [4].

Theorem 3.1. *Let the complete r'-partite graph $K_{p_1,p_2,...,p_{r'}} = K_{a_1\cdot p_1,a_2\cdot p_2,...,a_s\cdot p_s}$ on $n = a_1p_1 + a_2p_2 + \cdots + a_sp_s$ vertices with non-zero eigenvalues μ_i ($i = 1,2,...,s$) is Q-integral, where μ_i ($i = 1,2,...,s$) are all Q-eigenvalues of the nontrivial part of its Q-spectrum. Then complete r''-partite graph $K_{p_1,p_2,...,p_{r''}} = K_{b_1\cdot p_1,b_2\cdot p_2,...,b_s\cdot p_s}$ on n' vertices is Q-integral with non-zero Q-eigenvalues μ_i' ($i = 1,2,...,s$), for*

$$d_k = GCD\left(\prod_{i=2}^{s}(\mu_i - n + 2p_k), p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)\right), k = 1,2,...,s, \tag{5}$$

$$s_k = \frac{\prod_{i=2}^{s}(\mu_i - n + 2p_k)}{d_k}, k = 1,2,...,s, \tag{6}$$

$$r = LCM(r_1,r_2,....,r_s), r_k = \frac{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}{d_k}, k = 1,2,...,s, \tag{7}$$

$$b_k = a_k + \frac{s_k r}{r_k}t, k = 1,2,...,s, \tag{8}$$

$$\mu_1' = \mu_1 + 2rt, \mu_i' = \mu_i + rt, i = 2,...,s, \tag{9}$$

$$n' = n + rt, \tag{10}$$

for any positive integer t.

Proof. Using (2),(5-10) and Theorem 2.1 we have

$$b_k = a_k + r\frac{s_k}{r_k}t; k = 1,2,...,s,$$

$$b_k = a_k + r\frac{\prod_{i=2}^{s}(\mu_i - n + 2p_k)}{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}t; k = 1,2,...,s,$$

$$b_k = \frac{(\mu_1 - n + 2p_k)\prod_{i=2}^{s}(\mu_i - n + 2p_k) + rt\prod_{i=2}^{s}(\mu_i - n + 2p_k)}{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}; k = 1,2,...,s,$$

$$b_k = \frac{(\mu_1 - n + 2p_k + rt)\prod_{i=2}^{s}(\mu_i - n + 2p_k)}{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}; k = 1,2,...,s,$$

$$b_k = \frac{(2rt + \mu_1 - n - rt + 2p_k)\prod_{i=2}^{s}(rt + \mu_i - n - rt + 2p_k)}{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}; k = 1,2,...,s,$$

$$b_k = \frac{(\mu_1' - n' + 2p_k)\prod_{i=2}^{s}(\mu_i' - n' + 2p_k)}{p_k\prod_{i=1,i\neq k}^{s}2(p_k - p_i)}; k = 1,2,...,s.$$

From (6) and (7) follows that $b_k, k = 1,2,...,s$ are positive integers for every nonnegative integer t.

By Theorem 2.1 we have $-\infty < n - 2p_s < \mu_s < n - 2p_{s-1} < \mu_{s-1} < ... < n - 2p_1 < \mu_1 < +\infty$, from which we get $-\infty < n + rt - 2p_s < \mu_s + rt < n + rt - 2p_{s-1} < \mu_{s-1} + rt < ... < n + rt - 2p_1 < \mu_1 + rt < \mu_1 + 2rt < +\infty$. Using (9) we get $-\infty < n' - 2p_s < \mu_s' < n' - 2p_{s-1} < \mu_{s-1}' < ... < n' - 2p_1 < \mu_1' < +\infty$.

Now, by Theorem 2.1, the graph $K_{b_1\cdot p_1,b_2\cdot p_2,...,b_s\cdot p_s}$ is Q-integral. □

Theorem 3.2. *Let the complete r'-partite graph $K_{p_1,p_2,...,p_{r'}} = K_{a_1\cdot p_1,a_2\cdot p_2,...,a_s\cdot p_s}$ on $n = a_1p_1 + a_2p_2 + \cdots + a_sp_s$ vertices with non-zero eigenvalues ρ_i $(i = 1,2,...,s)$ is S-integral, where ρ_i $(i = 1,2,...,s)$ are all S-eigenvalues of the nontrivial part of its S-spectrum. Then complete r''-partite graph $K_{p_1,p_2,...,p_{r''}} = K_{b_1\cdot p_1,b_2\cdot p_2,...,b_s\cdot p_s}$ on n' vertices is S-integral with non-zero S-eigenvalues ρ'_i $(i = 1,2,...,s)$, for*

$$d_k = GCD\left(\prod_{i=2}^{s}(\rho_i - 2p_k + 1), p_k \prod_{i=1,i\neq k}^{s} 2(p_i - p_k)\right), k = 1,2,...,s, \tag{11}$$

$$s_k = \frac{\prod_{i=2}^{s}(\rho_i - 2p_k + 1)}{d_k}, k = 1,2,...,s, \tag{12}$$

$$r = LCM(r_1, r_2, ..., r_s), r_k = \frac{p_k \prod_{i=1,i\neq k}^{s} 2(p_i - p_k)}{d_k}, k = 1,2,...,s, \tag{13}$$

$$b_k = a_k + \frac{s_k r}{r_k} t, k = 1,2,...,s, \tag{14}$$

$$\rho'_1 = \rho_1 - rt, \rho'_i = \rho_i, i = 2,...,s, \tag{15}$$

for any positive integer t.

Proof. Proof is similar to the proof of theorem 3.1 and follows from theorem 2.2. □

From Theorems 2.3 and 2.4 we have the following corollary.

Corollary 3.3. *Let $G = K_{p_1,p_2,...,p_r}$. The following statements are equivalent:*
1. G is Q-integral.
2. G is S-integral.
3. G is QLS-integral.

From Corollary 3.3 and Corollary 2.9 of [14] we have the following corollary.

Corollary 3.4. *For any positive integer q, the complete multipartite graph $K_{a_1\cdot p_1 q,a_2\cdot p_2 q,...,a_s\cdot p_s q}$ is QLS-integral if and only if the graph $K_{a_1\cdot p_1,a_2\cdot p_2,...,a_s\cdot p_s}$ is QLS-integral.*

Remark 3.5. *Let $GCD(p_1,\dots,p_s)$ denote the greatest common divisor of the numbers $p_1,\dots,p_s$. Corollary 3.4 shows that it is reasonable to study QLS-integrality of graphs $K_{a_1\cdot p_1,a_2\cdot p_2,...,a_s\cdot p_s}$ only for $GCD(p_1,\dots,p_s) = 1$.*

4. Application of Main Results to Construction of New Classes of QLS-Integral Complete Multipartite Graphs

It is easy to see that for complete bipartite graphs we have the following results.

Corollary 4.1.
a. The graph K_{p_1,p_2} is QLS-integral for any positive integers p_1, p_2.
b. The graph K_{p_1,p_2} is AQLS-integral if and only if $p_1 \cdot p_2$ is a perfect square.

Table 1: QLS-integral complete graphs $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4}$.

No.	1	2	3	4	5	6	7	8
a_1	4	8	17	20	18	22	17	15
p_1	2	2	2	1	1	1	3	2
a_2	1	2	2	4	5	4	2	3
p_2	6	9	7	5	3	10	7	7
a_3	1	1	2	3	2	1	1	1
p_3	9	15	9	7	7	14	12	12
a_4	1	1	1	2	2	1	2	3
p_4	24	20	11	10	22	19	18	18
μ_1	71	117	143	151	161	171	209	213
μ_2	39	59	67	76	87	87	101	108
μ_3	32	44	61	69	80	69	92	97
μ_4	11	33	56	63	61	60	83	89
ρ_1	-25	-49	-67	-71	-71	-77	-97	-97
ρ_2	7	9	9	4	3	7	11	8
ρ_3	14	24	15	11	10	25	20	19
ρ_4	35	35	20	17	29	34	29	27
r	504	3432	9240	1260	1596	5928	1320	1320
b_1	$4 + 72t$	$8 + 528t$	$17 + 2244t$	$20 + 350t$	$18 + 399t$	$22 + 1672t$	$17 + 220t$	$15 + 198t$
b_2	$1 + 14t$	$2 + 104t$	$2 + 231t$	$4 + 63t$	$5 + 105t$	$4 + 247t$	$2 + 24t$	$3 + 36t$
b_3	$1 + 12t$	$1 + 44t$	$2 + 220t$	$3 + 45t$	$2 + 38t$	$1 + 57t$	$1 + 11t$	$1 + 11t$
b_4	$1 + 7t$	$1 + 39t$	$1 + 105t$	$2 + 28t$	$2 + 28t$	$1 + 52t$	$2 + 20t$	$3 + 30t$

Now, using computer search, Theorems 3.1,3.2 and Corollary 3.3 we construct new infinite classes of QLS-integral complete multipartite graphs $K_{b_1\cdot p_1, b_2\cdot p_2, \ldots, b_s\cdot p_s}$ where $s = 4, 5, 6$.

By computer we have found 8 Q-integral (S-integral) complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_4\cdot p_4}$ on less than 120 vertices (see also [8]). Their list together with the nontrivial part of their Q-spectrum (S-spectrum) is in the table 1, rows 2-17. Moreover, using theorems 3.1 and 3.2 and corollary 3.3 we can construct infinite classes of QLS-integral complete multipartite graphs $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4}$ for each of these graphs. The parameters b_i of these infinite classes are presented in table 1. Note that the non-trivial part of their Q-spectrum (S-spectrum) can be calculated using Theorems 3.1 and 3.2, formulas (9) and (15).

Corollary 4.2.

a. Let $a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4$ be those of Table 1. Then $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3, a_4\cdot p_4}$ is QLS-integral complete multipartite graph.

b. Let $b_1, p_1, b_2, p_2, b_3, p_3, b_4, p_4$ be those of Table 1. Then $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4}$ is QLS-integral complete multipartite graph for every $t \in N$.

Proof.

a. It is sufficient to use Theorems 2.1, 2.2 and 2.3.

b. It is sufficient to use Theorems 2.3, 3.1 and 3.2.

□

Using computer we have found 7 Q-integral (S-integral) complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3, a_4\cdot p_4, a_5\cdot p_5}$ on less than 1000 vertices (see also [8]). Their list together with the nontrivial part of their Q-spectrum (S-spectrum) is in Table 2, rows 2-21. Moreover, using Theorems 3.1, 3.2 and Corollary 3.3 we can construct infinite classes of QLS-integral complete multipartite graphs $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4, b_5\cdot p_5}$ for each of these graphs. The parameters b_i of these infinite classes are presented in table 2. Note that the non-trivial part of their Q-spectrum (S-spectrum) can be calculated using Theorems 3.1 and 3.2, formulas (9) and (15).

Table 2: QLS-integral complete graphs $K_{b_1 \cdot p_1, b_2 \cdot p_2, b_3 \cdot p_3, b_4 \cdot p_4, b_5 \cdot p_5}$.

No.	1	2	3	4	5	6	7
a_1	39	46	15	18	36	9	20
p_1	3	2	4	11	5	8	2
a_2	3	9	18	2	13	4	17
p_2	12	10	10	18	15	14	9
a_3	2	7	4	6	6	5	2
p_3	17	15	13	20	25	19	16
a_4	1	6	5	3	3	14	1
p_4	27	21	15	26	39	23	29
a_5	3	4	9	5	1	3	12
p_5	45	27	20	37	49	50	42
μ_1	655	1011	1067	1189	1341	1339	1454
μ_2	331	512	537	587	673	676	752
μ_3	319	497	523	580	649	665	730
μ_4	301	485	519	569	621	655	718
μ_5	280	471	512	553	596	607	698
ρ_1	-307	-491	-521	-573	-651	-645	-697
ρ_2	17	8	9	29	17	18	5
ρ_3	29	23	23	36	41	29	27
ρ_4	47	35	27	47	69	39	39
ρ_5	68	49	34	63	94	87	59
r	67320	1175720	240240	2217072	3403400	6588120	158340
b_1	$39 + 8415t$	$46 + 109480t$	$15 + 6825t$	$18 + 67184t$	$36 + 185640t$	$9 + 89838t$	$20 + 4524t$
b_2	$3 + 612t$	$9 + 20748t$	$18 + 8008t$	$2 + 7293t$	$13 + 65065t$	$4 + 39215t$	$17 + 3770t$
b_3	$2 + 396t$	$7 + 15827t$	$4 + 1760t$	$6 + 21736t$	$6 + 29172t$	$5 + 48300t$	$2 + 435t$
b_4	$1 + 187t$	$6 + 13260t$	$5 + 2184t$	$3 + 10659t$	$3 + 14025t$	$14 + 133672t$	$1 + 210t$
b_5	$3 + 510t$	$4 + 8645t$	$9 + 3861t$	$5 + 17160t$	$1 + 4550t$	$3 + 26565t$	$12 + 2436t$

Corollary 4.3.

a. Let $a_1, p_1, a_2, p_2, a_3, p_3, a_4, p_4, a_5, p_5$ be those of Table 2. Then $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3, a_4\cdot p_4, a_5\cdot p_5}$ is QLS-integral complete multipartite graph.

b. Let $b_1, p_1, b_2, p_2, b_3, p_3, b_4, p_4, b_5, p_5$ be those of Table 2. Then $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4, b_5\cdot p_5}$ is QLS-integral complete multipartite graph for every $t \in N$.

Proof. The proof is similar to that of corollary 4.2. □

Corollary 4.4.

a. Let $a_1 = 44, p_1 = 6, a_2 = 107, p_2 = 10, a_3 = 24, p_3 = 13, a_4 = 50, p_4 = 19, a_5 = 25, p_5 = 24, a_6 = 53, p_6 = 33$. Then $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3, a_4\cdot p_4, a_5\cdot p_5, a_6\cdot p_6}$ is QLS-integral complete multipartite graph. The nontrivial part of its Q-spectrum is $\{9847, 4932, 4921, 4915, 4901, 4889\}$ and the nontrivial part of its S-spectrum is $\{-4903, 12, 23, 29, 43, 55\}$.

b. Let $b_1 = 44 + 846032t$, $p_1 = 6$, $b_2 = 107 + 2054052t$, $p_2 = 10$, $b_3 = 24 + 460161t$, $p_3 = 13$, $b_4 = 50 + 956340t$, $p_4 = 19$, $b_5 = 25 + 477204t$, $p_5 = 24$, $b_6 = 53 + 1008007t$, $p_6 = 33$. Then $K_{b_1\cdot p_1, b_2\cdot p_2, b_3\cdot p_3, b_4\cdot p_4, b_5\cdot p_5, b_6\cdot p_6}$ is QLS-integral complete multipartite graph for every $t \in N$.

Proof. The proof is similar to that of corollary 4.2. The value of r is 94486392. □

Remark 4.5. *In [8] the following results for Q-integral graphs are given:*

1. (see Theorem 7 of [8]) The complete tripartite graph

$$K_{\frac{F_{2n}^2 - F_{2n}}{2}, \frac{F_{2n}^2 + F_{2n}}{2}, F_{2n}^2 - 1}$$

is Q-integral for $n \geq 2$, where $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$ are Fibonacci numbers.

2. (see Theorem 8 of [8]) The complete 4-partite graph $K_{3(b^2+1),(b^2+1)^2,9b^2,3b^2(b^2+1)}$ is Q-integral for any $b \in Z$.

Using Corollary 3.3 the graphs $K_{\frac{F_{2n}^2 - F_{2n}}{2}, \frac{F_{2n}^2 + F_{2n}}{2}, F_{2n}^2 - 1}$, $K_{3(b^2+1),(b^2+1)^2,9b^2,3b^2(b^2+1)}$ are QLS-integral. Moreover, using Theorems 3.1, 3.2 and Corollary 3.3 we can construct infinite classes of QLS-integral complete multipartite graphs for each of these graphs.

5. Conclusion

There are two ways of constructing infinite classes of QLS-integral complete r''-partite graphs for any QLS-integral r'-partite graph. One of them follows from Corollary 3.4. It keeps the number of partites and multiplies the number of vertices in each partite by q. The second method follows from Theorems 3.1, 3.2 and Corollary 3.3. It keeps the number of vertices in each partite and enlarges number of partites. Note that we can combine these methods.

In the paper new infinite families of QLS-integral complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_s\cdot p_s}$, where $s = 3, 4, 5, 6$ are given, what partly answers the questions 4.1 and 4.2 of [10] and also the questions 4.1 and 4.2 of [14]. However, when $s > 6$, we have not found such QLS-integral complete multipartite graphs, so the problem of existence of QLS-integral complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_s\cdot p_s}$ for $s > 6$ remain open. Thus we raise the following question.

Question 1. Are there any QLS-integral complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_s\cdot p_s}$ where $s > 6$?

The existence of QLS-integral complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_s\cdot p_s}$ where $a_1 = a_2 = \cdots = a_s = 1$ and $s < 5$ follows from [8] and this paper. Thus we raise the following question.

Question 2. Are there any QLS-integral complete multipartite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, \ldots, a_s\cdot p_s}$ where $a_1 = a_2 = \ldots = a_s = 1$ and $s \geq 5$?

It is easy to see that K_{p_1} and $K_{a_1\cdot p_1}$ are AQLS-integral for any $a_1, p_1 \in N$. Complete bipartite graphs K_{p_1,p_2} are AQLS-integral if and only if $p_1 \cdot p_2$ is a perfect square. Thus we give the following question.

Question 3. Are there any AQLS-integral complete multipartite graphs $K_{p_1,p_2,\ldots,p_s}$ where $p_1 < p_2 < \cdots < p_s$ and $s > 2$?

References

[1] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, D. Stevanović, A survey on integral graphs, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat., 13 (2002) 42-65.

[2] D. Cvetković, S. K. Simić, Towards a Spectral Theory of Graphs Based on the Signless Laplacian I., Publ. Inst. Math. (Beograd) 85 (99) (2009) 19–33.

[3] F. Harary, A.J.Schwenk, Which Graphs Have Integral Spectra? Graphs and Combinatorics, Lecture Notes in Mathematics 406, Springer-Verlag, Berlin (1974) 45–51.

[4] P. Híc, M. Pokorný, New Classes of Integral Complete n-Partite Graphs, Advances and Applications in Discrete Mathematics 7, No.2 (2011) 83–94.

[5] P. Híc, M. Pokorný, A Note on Integral Complete 4-partite Graphs, Discrete Math. 308 (2008) 3704-3705.

[6] P. Híc, M. Pokorný, P. Černek, New Sufficient Conditions for Integral Complete 3-partite Graphs, Appl. Anal. Discrete Math. 2 (2008) 276–284.

[7] S.M. Lv, L. Wei, H.X. Zhao, On the Seidel Integral Complete Multipartite Graphs, Acta Mathematicae Applicatae Sinica, English Series 28, No. 4 (2012) 705–710.

[8] M. Pokorný, P. Híc, D. Stevanović, Remarks on Q-integral Complete r-partite Graphs, Linear Algebra Appl. 439 (2013) 2029-2037.

[9] D. Stevanović, N.M.M. de Abreu, M.A.A. de Freitas, R. Del-Vecchio, Walks and Regular Integral Graphs, Linear Algebra Appl. 423 (2007) 119–135.

[10] L.G. Wang, G.P. Zhao, K.Li, Seidel Integral Complete r-Partite Graphs, Graphs and Combinatorics 30 (2) (2014) 479–493.

[11] L.G. Wang, X.L. Li, C. Hoede, Integral Complete r-partite Graphs, Discrete Math. 283 (2004) 231–241.

[12] L.G. Wang, X.D. Liu, Integral complete multipartite graphs, Discrete Math. 308 (2008) 3860–3870.

[13] L.G. Wang, Q. Wang, Integral complete multipartite graphs $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ with $s = 5, 6$, Discrete Math. 310 (2010) 812–818 .

[14] G.P. Zhao, L.G. Wang, K.Li, Q-integral complete r-partite graphs, Linear Algebra Appl. 438 (2013) 1067–1077.

[15] G.P. Zhao, L.G. Wang, The Laplacian polynomial of complete multipartite graphs, Fangzhi Gaoxiao Jichukexue Xuebao 24(2) (2011) 243–245 .